BE case: For the BE gas as $T \to 0$, a large number of particles condense in the ground state, $\varepsilon_{(gr)} = 0$, $\bar{n}_{(gr)} \simeq N$. Then from (10.3)

$$\overline{(\delta n_0)^2} = \overline{n^2} - (\bar{n})^2 \simeq \overline{n^2} - N^2, \qquad \text{...(10.26)}$$

and from (10.23)

$$\overline{(\delta n_0)^2} = \bar{n}_0(1+\bar{n}_0) \simeq N + N^2 \simeq N^2, \qquad \text{...(10.27)}$$

so that

$$\overline{n^2} \simeq 2N^2. \qquad \text{...(10.28)}$$

Instead of considering the single-particle state n_r, we can consider a group of g neighbouring states, all having the same mean occupation number $\bar{n}$. We can sum (10.23) over such a group of g neighbouring states containing $\bar{N} = g\bar{n}$ particles. The statistical independence of the probability distribution of the different single-particle states allows us to write

$$\overline{(\delta N)^2} = g\overline{(\delta n)^2} = g\bar{n}\left(1 \mp \bar{n}\right) = \bar{N}\left(1 \mp \frac{1}{g}\bar{N}\right) \qquad \text{...(10.29)}$$

The relation (10.23) is applicable to photons as well, even though (10.22) cannot be used, since $\mu = 0$ for photons.

We can use (10.29) for photons that obey BE statics, $n(\varepsilon) = (e^{\varepsilon/\theta} - 1)^{-1}$. The number of quantum states of the photons with frequencies between ν and $\nu + \Delta\nu$ is given by (4.118), $g = 8\pi V(\nu^2/c^3)\Delta\nu$. The total energy of the quanta in the frequency range is $E_{(\Delta\nu)} = Nh\nu$.

If we multiply (10.29) by $(h\nu)^2$,

$$\overline{(\delta E^{ph}_{(\Delta\nu)})^2} = h\nu E_{(\Delta\nu)} + \frac{c^3 (E_{\Delta\nu})^2}{8\pi V \nu^2 d\nu}. \qquad \text{...(10.30)}$$

This result was derived by Einstein. The first term on the right involving h is typical of the corpuscular nature of radiation. The second term, not involving h, represents the classical result for the energy fluctuations of black-body radiation. The result (10.30) implies that photons like to travel in bunches. Large photon density fluctuations have been experimentally observed.[1]

10.5 ONE-DIMENSIONAL RANDOM WALK

A drunk sailor, who has lost the sense of direction, takes a *random walk* in one dimension. Suppose he takes N steps of equal length l, each step being random (say) to the east or to the west. Each step has a probability $\frac{1}{2}$ of being in either direction. Let us find the probability that he is at a distance x from the starting point after such a walk.

[1]R.H. Brown and R.Q. Twiss, *Nature*, **177**, 27 (1956); E.M. Purcell, *Nature*, **178**, 1449 (1956).

Denote by $P(m, N)$ the probability that the sailor is at a point m steps away after N steps. The probability of any given sequence of N steps is $\left(\frac{1}{2}\right)^N$, because each step has a probability of $\frac{1}{2}$. Hence

$$P(m, N) = (\text{number of distinct sequences that reach } m \text{ after } N \text{ steps}) \times \left(\frac{1}{2}\right)^N.$$

To arrive at the point m, some set of $n_1 = \frac{1}{2}(N+m)$ steps out of N must be positive, and the remaining $n_2 = \frac{1}{2}(N-m)$ steps must be negative. Therefore, the number of distinct sequences that reach m is

$$W(m) = \frac{N!}{\left[\frac{1}{2}(N+m)\right]!\left[\frac{1}{2}(N-m)\right]!}, \qquad \text{...(10.31)}$$

and $$P(m, N) = \left(\frac{1}{2}\right)^N W(m). \qquad \text{...(10.32)}$$

For large N use the Stirling approximation in its more exact form (Appendix 2), $N! = (2\pi N)^{1/2} N^N e^{-N}$, or

$$\begin{aligned} \ln N! &= N \ln N - N + \frac{1}{2}\ln(2\pi N) \\ &= \left(N + \frac{1}{2}\right)\ln N - N + \frac{1}{2}\ln 2\pi . \end{aligned} \qquad \text{...(10.33)}$$

Then

$$\begin{aligned} \ln P(m, N) = {} & \left(N + \frac{1}{2}\right)\ln N - \frac{1}{2}(N + m + 1)\ln\frac{1}{2}(N+m) \\ & - \frac{1}{2}(N - m + 1)\ln\frac{1}{2}(N-m) - \frac{1}{2}\ln 2\pi - N\ln 2 . \end{aligned} \qquad \text{...(10.34)}$$

Since $m \ll N$, expand

$$\ln\left(1 \pm \frac{m}{N}\right) = \pm\frac{m}{N} - \frac{m^2}{2N^2} \pm \ldots \qquad \text{...(10.35)}$$

so that, using $\ln\frac{1}{2}(N \pm m) - \ln\frac{1}{2}N + \ln[1 \pm (m/N)]$,

$$\begin{aligned} \ln P(m, N) \simeq {} & \left(N + \frac{1}{2}\right)\ln N - \frac{1}{2}\ln 2\pi - N\ln 2 \\ & - \frac{1}{2}(N + m + 1)\left(\ln N - \ln 2 + \frac{m}{N} - \frac{m^2}{2N^2}\right) \end{aligned}$$

$$-\frac{1}{2}(N-m+1)\left(\ln N-\ln 2-\frac{m}{N}-\frac{m^2}{2N^2}\right)$$

$$\simeq -\frac{1}{2}\ln N+\ln 2-\frac{1}{2}\ln 2\pi-\frac{m^2}{2N}, \qquad ...(10.36)$$

or
$$P(m, N) \simeq \left(\frac{2}{\pi}N\right)^{1/2} \exp(-m^2/2N). \qquad ...(10.37)$$

As $x = ml$ and $m = n_1 - n_2 = n_1 - (N - n_1) = 2n_1 - N$, the probability that the sailor is between x and $x + dx$ after N steps is

$$P(x, N)dx = P(m, N)dm = P(m, N)\frac{dx}{2l}. \qquad ...(10.38)$$

We write $dx = 2l\ dm$, because m can take only integral values separated by an amount $\Delta m = 2$.

From (10.37, 38)

$$P(x, N)dx = (2\pi l^2 N)^{-1/2} \exp(-x^2/2Nl^2)dx. \qquad ...(10.39)$$

This is the *normal* or *Gaussian distribution*, which is usually written as

$$P(x) = (2\pi)^{-1/2}\gamma^{-1} \exp(-x^2/2\gamma^2), \quad \int_{-\infty}^{+\infty} P(x)dx = 1 \qquad ...(10.40)$$

It has a symmetrical peak situated at $x = 0$. The width of the peak increases with γ(Fig. 3.8).

To introduce time, we assume that the sailor takes $N = nt$ steps in time t. Then the probability of the sailor being in the interval dx at x after time t is

$$P(x)dx = (2\pi l^2 nt)^{-1/2} \exp(-x^2/2l^2 nt)dx. \qquad ...(10.41)$$

The mean square distance travelled is given by the mean square fluctuation

$$\overline{(\delta x)^2} = \overline{x^2} = \int_{-\infty}^{+\infty} x^2 P(x)dx = l^2 nt = \gamma^2, \qquad ...(10.42)$$

where we have used

$$\int_{-\infty}^{+\infty} x^2 \exp(-ax^2)dx = \frac{1}{2}\left(\frac{\pi}{a^3}\right)^{1/2}.$$

If τ is the time taken for each step, the $t = \tau N$ and $1/\tau = v$ is the velocity. We can write the *conditional probability* that the sailor will be within dx at x at time t if he was at $x = 0$ at $t = 0$, as

$$P(0, 0; x, t) = (4\pi Dt)^{-1/2} \exp(-x^2/4Dt)dx, \quad D = \frac{1}{2}v^2\tau. \qquad ...(10.43)$$

Note that $n\tau = 1$. The spread of the distribution increases with t, and

$$\overline{x^2} = (vt)^2 N = 2Dt. \qquad ...(10.44)$$

D is the particle diffusion constant.

The problem of N particles, each having a magnetic moment μ which may be either parallel or antiparallel to a magnetic field H was discussed in Sec. 3.8. The calculation of the probability distribution of the total magnetic moment M for $H = 0$ is identical with that in the random walk problem [compare (10.32, 37) with (3.94)]. If we write $M = m\mu_H$, then (10.37) gives for the entropy

$$\sigma = \ln P(m, N) \simeq \text{constant} - \frac{m^2}{2N}. \quad ...(10.45)$$

In the presence of the magnetic field H,

$$E = -m\mu_H H, \quad F = E - kt\sigma \simeq -m\mu_H H + \frac{m^2 kT}{2N} + \text{constant}.$$

If F is minimum, $\partial F/\partial m = 0$ gives

$$\frac{m}{N} = \frac{\mu_H H}{kT}, \quad ...(10.46)$$

$$M = m\mu_H \simeq \frac{N\mu_H^2 H}{kT}. \quad ...(10.47)$$

Apart from numerical factors and replacement of kT by $\varepsilon_F(0)$, (10.47) agrees with (7.45).

10.6 RANDOM WALK[2] AND BROWNIAN MOTION

A very small particle immersed in a liquid exhibits a random type of motion. It is called *Brownian motion*. It is produced by the thermal fluctuation of pressure on the particle. Because of the fluctuations, the forces do not always cancel and the particle is knocked about in a random way.

The Brownian motion in one dimension is like a random walk along a line. At the end of each period of time τ the particle has either moved a distance $l = v\tau$ to the right or a distance l to the left. If the direction of each successive step is a *random variable*, then the probability that during $+N$ periods the particle has made s positive and $N - s$ negative steps, resulting in net displacement $x_s = [s - (N - s)]l = (2s - N)l$, is

$$P_s(N) = \frac{N!}{s!(N-s)!} P^s Q^{N-s}. \quad ...(10.48)$$

It is called the binomial distribution (Appendix 1) and reduces to (10.32) for

$$P = 1 - Q = \frac{1}{2} \text{ and } m = 2s - N. \text{ By definition}$$

$$\bar{x}_s = \sum_{s=0}^{N} x_s P_s(N) = \sum_{s=0}^{N} (2_s - N) l P_s(N), \quad ...(10.49)$$

$$\overline{(x_s - \bar{x}_s)^2} = \sum_{s=0}^{N} (x_s - \bar{x}_s)^2 P_s(N) = \sum_{s=0}^{N} [(2s - N)l - \bar{x}_s]^2 P_s(N). \quad ...(10.50)$$

[2] For a detailed survey see, for example, S. Chandrasekhar, *Rev. Mod. Phys.* **15**, 1 (1943).